NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

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Reflection from Inhomogeneous Plane Stratified Media II.

IRVIN KAY

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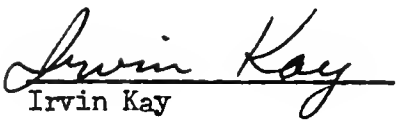

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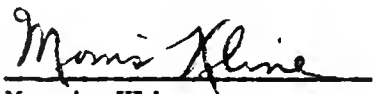

Morris Kline
Project Director

March, 1958

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Abstract

This report, as the title indicates, is an extension of the work begun in Research Report No. WP-1.

The approximation method described in WP-1 is applied here to the simple problem of reflection from a plane interface separating two homogeneous media with different dielectric constants. The resulting approximate reflection coefficient is the first term of a rapidly convergent series whose sum is shown to be the standard exact expression for the reflection coefficient of this problem.

In the second part of the present work a different approximation method is described. The new method is appropriate for certain cases in which the previous method fails or gives poor results, e.g., the case of a wave incident at a nearly grazing angle.

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1. Introduction

In Research Report WP-1* a method for obtaining the approximate reflection coefficient of an inhomogeneous transmission line was described. An estimate of the error produced by the approximation was given, and it was seen that the error depended only on a quantity which was called the total variation of the characteristic impedance throughout the medium. Thus, we refer to the approximation of report WP-1 as the small variation approximation.

A basic assumption in the derivation of the small variation approximation was that the characteristic impedance be a continuous function of distance along the line. Since the error estimate apparently has nothing to do with the continuity of the characteristic impedance, we might expect that the method can be extended to cover a more general class of problems. Indeed, we shall demonstrate in Section 2 of this report that the characteristic impedance is allowed to have jump discontinuities, and we shall indicate how the result of report WP-1 can be extended to cover the case of an arbitrary medium whose characteristic impedance is piecewise continuous. A simple example will be treated in detail to illustrate the application of the small variation approximation as well as the significance of our error estimate.

In Section 3 we shall describe another class of approximations for the wave solutions of an inhomogeneous transmission line. The new class of approximations will apply in some cases to which the small variation approximation cannot be applied: in particular, the case of a plane wave incident at an angle near ninety degrees on an inhomogeneous plane stratified medium. An example of the new method applied to this problem will be given in Section 4.

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* Kay, Irvin - Reflection from inhomogeneous plane stratified media, I; N.Y.U., IMS, Div. EM Res., Research Report WP-1, January, 1958.

It can be seen from this description that Section 2 is independent of Sections 3 and 4, except insofar as all three sections refer to the results of report WP-1.

2. Example of the small variation perturbation

In order to show in detail the exact nature of the small variation approximation we shall demonstrate its application to a very simple problem whose solution is well known. The example we shall use devolves upon the case of two different homogeneous transmission lines joined at the point of discontinuity $z = 0$, the line in the region $z < 0$ having the constant characteristic impedance ζ_1 and the line in the region $z > 0$ having the constant characteristic impedance ζ_2.

The well-known exact relation for the reflection coefficient of this pair of transmission lines at the point $z = 0$ is

$$R = (\zeta_2 - \zeta_1)/(\zeta_2 + \zeta_1). \tag{2.1}$$

It is possible to derive (2.1) from the integral equations of report WP-1:

$$u(z) = u(\Delta) + \int_z^\Delta \left\{\zeta'(y)/2\zeta(y)\right\} \exp\left\{-2j\int_y^\Delta k(s)ds\right\} d(y)dy \tag{2.2}$$

$$d(z) = \int_z^\Delta \left\{\zeta'(y)/2\zeta(y)\right\} \exp\left\{2j\int_y^\Delta k(s)ds\right\} u(y)dy$$

for the upgoing and downgoing factors $u(z)$ and $d(z)$ of the traveling waves; the reflection coefficient at the point $z = 0$ is given in terms of these functions by

$$R = d(0)/u(0). \tag{2.3}$$

In report WP-1 we derived equations (2.2) under the assumption that the characteristic impedance $\zeta(z)$ is a continuous function, whereas in the present case $\zeta(z)$ has a jump discontinuity at $z = 0$. Thus, it will be necessary to treat the transmission line of our example as the limiting form of a sequence of continuous transmission lines. This procedure is also obviously necessitated by the presence of the derivative of $\zeta(z)$ in both integrals on the right of (2.2). Each one of the continuous transmission lines in our assumed sequence is homogeneous except in an interval $0 \leq z \leq \delta$ where $\zeta(z)$ varies continuously from ζ_1 to ζ_2. In the limit $\delta \to 0$ the sequence converges to the case of our example.

If we carry out the infinite iteration of (2.2) described in report WP-1, the iteration whose first order term constitutes the approximate solution of the small variation method, we shall obtain the functions $u(z)$ and $d(z)$ exactly and thus from (2.3) the exact reflection coefficient at $z = 0$. The infinite iteration is too complicated to apply exactly to any one of the sequences of continuous transmission line problems. However, we can obtain an estimate from which the limiting form of the sequence of solutions corresponding to the sequence of continuous transmission lines can be obtained. The limit of the resulting sequence of reflection coefficients will turn out to be identical with the reflection coefficient given by (2.1).

The infinite iteration of the system (2.2) starting with $u_o = u(\Delta)$, $d_o = 0$

$$u(z) = u(\Delta) + u(\Delta) \sum_{n=1}^{\infty} \int_z^{\Delta} dy_{2n} \sigma(y_{2n}) e^{-1}(y_{2n}) \int_{y_{2n}}^{\Delta} dy_{2n-1} \sigma(y_{2n-1}) e(y_{2n-1}) \int_{y_{2n-1}}^{\Delta} \cdots \int_{y_2}^{\Delta} dy_1 \sigma(y_1) e(y_1)$$

(2.4)

$$d(z) = u(\Delta) \sum_{n=1}^{\infty} \int_z^{\Delta} dy_{2n-1} \sigma(y_{2n-1}) e(y_{2n-1}) \int_{y_{2n-1}}^{\Delta} dy_{2n-2} \sigma(y_{2n-2}) e^{-1}(y_{2n-2}) \int_{y_{2n-2}}^{\Delta} \cdots \int_{y_2}^{\Delta} dy_1 \sigma(y_1) e(y_1),$$

where $\sigma(y) = \zeta'(y)/2\zeta(y)$ and $e(y) = \exp\left[2j\int_y^{\Delta} k(s)ds\right]$. To compute the reflection coefficient from (2.4) we must first set $z = 0$ and take into account the fact that $\sigma(y) = 0$ for all $y > \delta$ since $\zeta(y)$ is constant when $y > \delta$. Thus, the outside integrals go from 0 to δ, and the inside integrals all have δ for their upper limits.

Using the fact that

$$\int_0^y \sigma(s)ds = \log\left[\zeta(y)/\zeta(0)\right]^{1/2}, \tag{2.5}$$

we can integrate the integrals in (2.4) by parts starting from the outer integral in each term. Thus we have, using (2.5),

$$J = \int_0^{\delta} dy_n \sigma(y_n)e(y_n) \int_{y_n}^{\delta} dy_{n-1} \sigma(y_{n-1})e^{-1}(y_{n-1}) \int_{y_{n-1}}^{\delta} \cdots \int_{y_2}^{\delta} dy_1 \sigma(y_1)e(y_1)$$

$$= e(y_n)\log\left[\zeta(y_n)/\zeta(0)\right]^{1/2} \cdot \int_{y_n}^{\delta} dy_{n-1} \sigma(y_{n-1})e^{-1}(y_{n-1}) \int_{y_{n-1}}^{\delta} \cdots \int_{y_2}^{\delta} dy_1 \sigma(y_1)e(y_1) \Big|_{y_n}^{\delta} = 0$$

$$+ \int_0^{\delta} dy_{n-1} e(y_{n-1})\log\left[\zeta(y_{n-1})/\zeta(0)\right]^{1/2} \cdot \sigma(y_{n-1})e^{-1}(y_{n-1}) \int_{y_{n-1}}^{\delta} \cdots \int_{y_2}^{\delta} dy_1 \sigma(y_1)e(y_1)$$

$$+ O(\delta),$$

where the first term on the right of the equation for J vanishes. The term $O(\delta)$ refers to the integral of a bounded function between the limits 0 and δ. The integral J can then be written:

$$J = \int_0^{\delta} dy_{n-1}\sigma(y_{n-1})\log\left[\zeta(y_{n-1})/\zeta(0)\right]^{1/2}\int_{y_{n-1}}^{\delta}\cdots\int_{y_2}^{\delta} dy_1\sigma(y_1)e(y_1) + O(\delta)$$

$$= (1/2)\int_0^{\delta} dy_{n-1}d\left\{\log^2\left[\zeta(y_{n-1})/\zeta(0)\right]^{1/2}\right\}\Big/dy\int_{y_{n-1}}^{\delta}\cdots\int_{y_2}^{\delta} dy_1\sigma(y_1)e(y_1)+O(\delta).$$

We integrate the outer integral by parts, again applying the same sort of discussion, and obtain

$$J = (1/3!)\int_0^{\delta} dy_{n-2}e(y_{n-2})d\left\{\log^3\left[\zeta(y_{n-2})/\zeta(0)\right]^{1/2}\right\}\Big/dy_{n-2}\int_{y_{n-2}}^{\delta}\cdots\int^{\delta} dy_1\sigma(y_1)ey_1+ O(\delta).$$

Continuing this process, we finally arrive at the result

$$J = (1/n!)\log^n\left[\zeta(\delta)/\zeta(0)\right]^{1/2} + O(\delta). \tag{2.6}$$

The term $O(\delta)$ in (2.6) is the sum of integrals of bounded functions between 0 and δ, and each term of this sum is also of order $1/(n-1)!$ by arguments similar to those just used here and in report WP-1.

We now substitute (2.6) into (2.4), remembering our definition of J, and obtain

$$u(\delta) = u(\Delta) + u(\Delta)\sum_{n=1}^{\infty}\left[1/(2n)!\right]\log^{2n}\left[\zeta(\delta)/\zeta(0)\right]^{1/2} + O(\delta) \tag{2.7}$$

$$d(\delta) = u(\Delta)\sum_{n=1}^{\infty}\left[1/(2n-1)!\right]\log^{2n-1}\left[\zeta(\delta)/\zeta(0)\right]^{1/2} + O(\delta),$$

where we have used the fact that $\zeta(y)$ is constant for $y > \delta$.

The quantity $\zeta(0)$ of equations (2.7) is equal to ζ_1 of equation (2.1) and the quantity $\zeta(\delta)$ of (2.7) is equal to ζ_2 of (2.1). The term $O(\delta)$ occurring in both of equations (2.7) approaches zero in the limit as δ approaches zero since $O(\delta)$ is the sum of an infinite series, the nth term of which is $O[\delta/(n-1)!]$. Therefore, (2.7) has the limiting form

$$u(0^+) = \sum_{n=0}^{\infty} \left[1/(2n)!\right] \log^{2n}\left[\zeta_2/\zeta_1\right]^{1/2} = 2\cosh\left[\log(\zeta_2/\zeta_1)^{1/2}\right] \tag{2.8}$$

$$d(0^+) = \sum_{n=1}^{\infty} \left[1/(2n-1)!\right] \log^{2n-1}\left[\zeta_2/\zeta_1\right]^{1/2} = 2\sinh\left[\log(\zeta_2/\zeta_1)^{1/2}\right].$$

After substituting (2.8) into (2.3), we obtain

$$\begin{aligned} R &= \sinh(\zeta_2/\zeta_1)^{1/2} / \cosh(\zeta_2/\zeta_1)^{1/2} \\ &= \left[(\zeta_2/\zeta_1)^{1/2} - (\zeta_1/\zeta_2)^{1/2}\right] / \left[(\zeta_2/\zeta_1)^{1/2} + (\zeta_1/\zeta_2)^{1/2}\right] \\ &= \left[\zeta_2 - \zeta_1\right] / \left[\zeta_2 + \zeta_1\right], \end{aligned} \tag{2.9}$$

which is the same as (2.1).

The small variation method of report WP-1 gives the first term of the cosh series in (2.8) for $u(0)$ and the first term of the sinh series in (2.8) for $d(0)$ so that

$$R \sim \log(\zeta_2/\zeta_1)^{1/2}. \tag{2.10}$$

The method of this example can be used to apply the small variation method to the reflection problem for an arbitrary piecewise continuous medium. Such a problem must again be considered to be the limiting case of a sequence

of problems for which the medium is continuous. Thus, suppose as usual that $\zeta(z)$ is constant for $z > \Delta$, and suppose that $\zeta(z)$ has a finite jump discontinuity at each of the points z_i,

$$z_i < z_{i+1}, \quad 0 < z_i < \Delta, \quad i = 1, \ldots, n,$$

where n is the number of discontinuities of $\zeta(z)$ in the interval $0 \leq z \leq \Delta$. We have for the first order approximation to the reflection coefficient in the case of a continuous $\zeta(z)$:

$$(2.11) \qquad R \sim \int_0^{\Delta} \left\{ \zeta'(y)/2\zeta(y) \right\} \exp\left\{ 2j \int_y^{\Delta} k(s)ds \right\} dy$$

$$= \left[\int_0^{z_1-\delta} + \sum_{i=1}^{n-1} \int_{z_i+\delta}^{z_{i+1}-\delta} + \int_{z_n+\delta}^{\Delta} \right] \left\{ \zeta'(y)/2\zeta(y) \right\} \exp\left\{ 2j \int_y^{\Delta} k(s)ds \right\} dy$$

$$+ \sum_{i=1}^{n} \int_{z_i-\delta}^{z_i+\delta} \left\{ \zeta'(y)/2\zeta(y) \right\} \exp\left\{ 2j \int_y^{\Delta} k(s)ds \right\} dy.$$

Now we consider the problem P_o with the discontinuous $\zeta(z)$ to be the limiting case of a sequence of problems P_δ. In every P_δ, $\zeta(z)$ is continuous and in particular goes continuously from its value at $z_i-\delta$ to its value at $z_i+\delta$ in the neighborhood of each point z_i which is a point of discontinuity of the discontinuous $\zeta(z)$ of our original problem P_o. We assume that the characteristic impedance $\zeta(z)$ of each problem P_δ of our sequence is equal to that of problem P_o at every point outside the intervals

$$z_i - \delta < z < z_i + \delta, \quad i = 1, \ldots, n$$

about the points of discontinuity z_i, $i = 1, \ldots, n$. We expect the sequence of reflection coefficients R_δ corresponding to the problems P_δ to approach the reflection coefficient R_o of problem P_o in the limit as δ approaches zero. To evaluate this limit we first integrate (2.11), by parts, as in our previous example. We have in the limit $\delta \to 0$:

$$(2.12)\quad R \sim \left\{\log \zeta^{1/2}(z_1)\right\}\exp\left\{2j\int_{z_1}^{\Delta} k(s)ds\right\} - \left\{\log \zeta^{1/2}(0)\right\}\exp\left\{2j\int_{o}^{\Delta} k(s)ds\right\}$$

$$+ \sum_{i=1}^{n-1}\left[\left\{\log \zeta^{1/2}_{-}(z_{i+1})\right\}\exp\left\{2j\int_{z_{i+1}}^{\Delta} k(s)ds\right\} - \left\{\log \zeta^{1/2}_{+}(z_i)\right\}\exp\left\{2j\int_{z_i}^{\Delta} k(s)ds\right\}\right]$$

$$+ \sum_{i=1}^{n}\left\{\log \zeta^{1/2}_{+}(z_i) - \log \zeta^{1/2}_{-}(z_i)\right\}\exp\left\{2j\int_{z_i}^{\Delta} k(s)ds\right\}$$

$$+ j\int_{o}^{\Delta} k(y)\left\{\log \zeta(y)\right\}\exp\left\{2j\int_{y}^{\Delta} k(s)ds\right\}dy.$$

By combining terms in (2.12) we arrive at

$$(2.13)\quad R \sim \left\{\log \zeta^{1/2}_{+}(z_n)\right\}\exp\left\{2j\int_{z_n}^{\Delta} k(s)ds\right\} - \left\{\log \zeta^{1/2}_{-}(0)\right\}\exp\left\{2j\int_{o}^{\Delta} k(s)ds\right\}$$

$$+ j\int_{o}^{\Delta} k(y)\left\{\log \zeta(y)\right\}\exp\left\{2j\int_{y}^{\Delta} k(s)ds\right\}dy.$$

The formula (2.13) is completely general for media with piecewise continuous $\zeta(z)$. Since no derivatives appear in (2.13), this result can be applied

directly to physical data without making any assumptions about the derivative of $\zeta(z)$ which cannot be measured in practice.

3. Perturbation about a standard transmission line

Since there are occasions when the error in the small variation method of report WP-1 is not too large, it may be useful to consider another more general perturbation method. If we can calculate the wave solutions of some standard transmission line whose electrical properties do not differ much from the one in which we are actually interested, we can devise a perturbation scheme for obtaining approximate wave solutions for the actual transmission line of interest in terms of the solutions for the standard line. A convenient choice for a standard transmission line from which we can construct a perturbation solution is, for example, a homogeneous line whose constant electrical parameters L and C are perhaps average values of the $L(z)$ and $C(z)$ of the actual line which may be inhomogeneous. Some of the advantages of the small variation method will not be present in the new method; e.g., the error estimate of our former approximation was independent of frequency, as we observed in report WP-1, whereas the error in the present approximation will depend upon the frequency in general. The present method can then be expected to provide a more appropriate solution to the inhomogeneous transmission line problem, only in the highly specialized situation for which it is designed in a given case.

We start, as before, with the problem of obtaining the wave solutions $V(z)$ and $I(z)$ of an inhomogeneous transmission line system

$$dV(z)/dz = - j\omega I(z)$$
$$dI(z)/dz = - j\omega C(z)V(z), \tag{3.1}$$

where we have chosen units of length so that $L(z) = 1$ in (3.1). At some point

$z = \Delta$, a load impedance Z or load admittance $Y = 1/Z$ is given so that

$$I(\Delta) = YV(\Delta). \tag{3.2}$$

Let us assume that we have available the complete set of linearly independent solutions of a system

$$\begin{aligned} dv(z)/dz &= - j\omega i(z) \\ di(z)/dz &= - j\omega \mathcal{L}(z) i(z); \end{aligned} \tag{3.3}$$

i.e., we have a non-singular solution matrix

$$\begin{pmatrix} v_1(z) & v_2(z) \\ i_1(z) & i_2(z) \end{pmatrix} = (v) \tag{3.4}$$

which satisfies

$$d(v)/dz = - j\omega(K)(v), \tag{3.5}$$

where $(K) = \begin{pmatrix} 0 & 1 \\ \mathcal{L}(z) & 0 \end{pmatrix}$.

It is a well-known fact that the matrix (v) of (3.4) has a constant determinant for all z. This determinant is fixed arbitrarily by properly choosing the initial values of the independent solutions which form its columns. We can therefore require that

$$\det(v) = 1. \tag{3.6}$$

Relation (3.6) implies that the inverse matrix of (3.4) is given by

$$(v)^{-1} = \begin{pmatrix} i_2 & -v_2 \\ -i_1 & v_1 \end{pmatrix}. \tag{3.7}$$

If we let E be the vector $\begin{pmatrix} V \\ I \end{pmatrix}$, (C) the matrix $\begin{pmatrix} 0 & 1 \\ C & 0 \end{pmatrix}$ and $(\mathcal{T})$ the matrix (C) - (K), we can write (3.1) in the form

$$dE/dz = - j\omega(C)E = - j\omega(K)E - j\omega(\mathcal{T})E. \tag{3.8}$$

The equation (3.8) has a form suitable for applying the perturbation method, perturbing about the standard transmission line of (3.5). To carry out the process we write

$$E = (v)\hat{E}, \tag{3.9}$$

where, for the components of E, we have

$$\hat{E} = \begin{pmatrix} \hat{V} \\ \hat{I} \end{pmatrix}.$$

On substituting (3.9) into (3.8), making use of (3.5), we obtain

$$d\hat{E}/dz = - j\omega(v)^{-1}(\mathcal{T})(v)\hat{E}. \tag{3.10}$$

In order to complete the perturbation procedure we shall form an equivalent system of integral equations for the components of $\hat{E}$. However, to obtain the most rapidly convergent iteration sequence we can, without complicating the final expressions, we should remove the diagonal part of the matrix $(v)^{-1}(\mathcal{T})(v)$ in (3.10) and further normalize the still somewhat arbitrary matrix (v) so that one component of the initial vector in our final system of integral equations is zero. As we have observed in Section 2, and in report WP-1, a system of integral equations, with an initial vector having one component zero and an anti-diagonal kernel matrix, will produce iteration series for the components of the solution vector such that each series will be lacking every other term.

With this principle in mind, we take (σ) the diagonal part of the matrix $(v)^{-1}(\Gamma)(v)$ of (3.10), and define a diagonal matrix (τ) as a solution of the differential equation

$$d(\tau)/dz = - j\omega(\sigma)(\tau). \tag{3.11}$$

We then define $\tilde{E}$ by

$$\hat{E} = (\tau)\tilde{E}. \tag{3.12}$$

After substituting (3.12) into (3.10) we obtain, using (3.11)

$$d\tilde{E}/dz = - j\omega(\tau)^{-1}(\gamma)(\tau)\tilde{E}, \tag{3.13}$$

where (γ) is $(v)^{-1}(\Gamma)(v)-(\sigma)$, and we can write finally:

$$d\tilde{E}/dz = - j\omega(G)\tilde{E},$$

where $(G) = (\tau)^{-1}(\gamma)(\tau)$ is an anti-diagonal matrix, as required.

If we choose initial conditions on the matrix (v) of (3.4) in such a way that, in addition to $\det(v) = 1$, we have

$$i_1(\Delta) = Yv_1(\Delta), \tag{3.14}$$

the component $\tilde{I}(\Delta)$ of the vector $\tilde{E}$ at $z = \Delta$ will vanish; i.e., we shall have

$$\tilde{E}(\Delta) = \begin{pmatrix} \tilde{V}(\Delta) \\ 0 \end{pmatrix}, \tag{3.15}$$

as required.

Before carrying out the last step in deriving our approximate wave solution for the transmission line given by (3.1), let us set down explicitly the expressions for the various matrices which occur in this derivation. We have, writing

(3.16) $$C(z) - \mathcal{H}(z) = Q(z),$$

the matrices:

(3.17) $$(\mathcal{T}) = \begin{pmatrix} 0 & 0 \\ Q(z) & 0 \end{pmatrix},$$

(3.18) $$(\sigma) = Q(z)v_1(z)v_2(z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(3.19) $$(\tau) = \begin{pmatrix} \exp\left\{-j\omega\int_z^\Delta Q(y)v_1(y)v_2(y)dy\right\} & 0 \\ 0 & \exp\left\{j\omega\int_z^\Delta Q(y)v_1(y)v_2(y)dy\right\} \end{pmatrix}$$

(3.20) $$(G) = Q(z)\begin{pmatrix} 0 & -v_2^2(z)\exp\left\{2j\omega\int_z^\Delta v_1(y)v_2(y)Q(y)dy\right\} \\ v_1^2(z)\exp\left\{-2j\omega\int_z^\Delta v_1(y)v_2(y)Q(y)dy\right\} & 0 \end{pmatrix}$$

After an integration (3.13) becomes

(3.21) $$\tilde{E}(z) = \tilde{E}(\Delta) + j\omega\int_z^\Delta (G)\tilde{E}(y)dy.$$

This leads to the first order approximation obtained by a single iteration,

(3.22) $$\tilde{V}(z) \sim \tilde{V}(\Delta)$$

$$\tilde{I}(z) \sim \tilde{V}(\Delta)\, j\omega \int_z^\Delta Q(y)v_1^2(y)\exp\left\{-2j\omega\int_y^\Delta Q(s)v_1(s)v_2(s)ds\right\}dy.$$

The error of the approximation (3.22) can be estimated by the expressions (4.12) and (4.4) of report WP-1. The function $\mu(z)$, given by (4.4), is in the present case

$$\mu(z) = \omega|Q(z)|\left\{|v_1(z)|^4 + |v_2(z)|^4\right\}^{1/2} \tag{3.23}$$

providing the phase $\int_z^{\Delta} v_1(y)v_2(y)Q(y)dy$ is real. The error given by (4.12) of report WP-1 for $\tilde{V}(z)$ is

$$\varepsilon_v \leq \exp\int_z^{\Delta}\mu(y)dy - \left\{1 + \int_z^{\Delta}\mu(y)dy\right\} \tag{3.24}$$

and for $\tilde{I}(z)$

$$\varepsilon_I \leq \exp\int_z^{\Delta}\mu(y)dy - \left\{1 + \int_z^{\Delta}\mu(y)dy + \left[\int_z^{\Delta}\mu(y)dy\right]^2/2\right\}.$$

These expressions follow from the fact that each of the two iteration series skips every other term so that the next term of the series for $\tilde{V}(z)$ is of second order and the next term of the series for $\tilde{I}(z)$ is of third order.

4. Example:

As an example of the perturbation method described in Section 3 we shall consider the case which arises when a plane wave is incident at nearly grazing incidence on a stratified medium. It is convenient in this case to perturb about a medium with $C(z) \equiv 0$. Thus, $\mathcal{H}(z) \equiv 0$, and we have for the equations (3.2)

$$dv/dz = -j\omega i \tag{4.1}$$

$$di/dz = 0.$$

The general solution of (4.1) has the form

$$i = \text{constant} \tag{4.2}$$

$$v = -j\omega iz + v(0).$$

We must choose appropriate values of i and v(0) to form a solution matrix whose determinant has the value one and so that (3.14) holds. Moreover, in order to minimize the error, the remaining constants should be chosen so that the phase of the integrand of (3.22) is real; i.e., the product $v_1(z)v_2(z)$ should be real. A choice of constants which satisfies these requirements gives for the elements of the solution matrix

$$\begin{pmatrix} v_1(z) & v_2(z) \\ i_1(z) & i_2(z) \end{pmatrix}$$

of (3.3) the values

$$v_1(z) = \lambda \left\{ 1 - \omega^2 Y^2 \Delta\ (z-\Delta) - j\omega Yz \right\} \Big/ (1 + \omega^2 Y^2 \Delta^2) \tag{4.3}$$

$$v_2(z) = -\left\{ 1 - \omega^2 Y^2 \Delta\ (z-\Delta) + j\omega Yz \right\} \Big/ 2\lambda Y$$

$$i_1(z) = \lambda Y \Big/ (1 + j\omega Y\Delta)\ (= \text{constant})$$

$$i_2(z) = (1 + j\omega Y\Delta) \Big/ 2\lambda\ (= \text{constant}),$$

where the constant λ is still arbitrary.

Since λ is arbitrary it can be chosen so that the quantity $\mu(z)$ of (3.23) which appears in the error estimate (3.24) is a minimum. We observe that except for constant factors $\mu(z)$ has the form

$$\mu(z) = (\alpha\lambda^4 + \beta\lambda^{-4})^{1/2}, \tag{4.4}$$

from the form of the expressions for $v_1(z)$ and $v_2(z)$ in (4.3). By differentiating (4.4) with respect to λ and setting the resulting expression equal to zero, we obtain for the minimizing value of λ,

$$\lambda = (\beta/\alpha)^{1/8}. \tag{4.5}$$

We compare (4.4) with (3.23), taking into account (4.3), and obtain from (4.5)

$$\lambda = (1 + \omega^2 Y^2 \Delta^2)^{1/2} \Big/ (2Y)^{1/2}. \tag{4.6}$$

Observe that fortuitously the minimizing λ of (4.6) is independent of z, as it must be if it is to be of any use, since we can only allow a constant λ in (4.3).

In the present example the function $Q(z)$ is given by

$$Q(z) = C(z). \tag{4.7}$$

From the following expressions the approximate wave solutions of (3.1), subject to the load admittance Y at the point $z = \Delta$, can be obtained:

$$\begin{aligned} &\tilde{V}(z) \sim \tilde{V}(\Delta) \\ &\tilde{I}(z) \sim \tilde{V}(\Delta) \int_z^{\Delta} C(y) v_1^2(y) \exp\left\{2j\omega \int_y^{\Delta} C(s) |v_1(s)|^2 ds\right\} dy, \end{aligned} \tag{4.8}$$

where

$$v_1(z) = \left\{1 - \omega^2 Y^2 \Delta (z - \Delta) - j\omega Y z\right\} \Big/ \left\{2Y(1 + \omega^2 Y^2 \Delta^2)\right\}^{1/2}. \tag{4.9}$$

To obtain $V(z)$ and $I(z)$ we combine (3.9) and (3.12):

$$E = (v)(\gamma)\tilde{E}.$$

We then have

(4.10)

$$V(z) \sim \tilde{V}(\Delta)\left[v_1(z)\exp\left\{j\omega\int_z^\Delta C(y)|v_1(y)|^2dy\right\}\right.$$

$$\left. - j\omega\bar{v}_1(z)\int_z^\Delta C(y)v_1^2(y)\exp\left\{2j\omega\int_y^\Delta C(s)|v_1(s)|^2ds\right\}dy\ \exp\left\{-j\omega\int_z^\Delta C(y)|v_1(y)|^2dy\right\}\right]$$

$$I(z) \sim \tilde{V}(\Delta)(Y/2)^{1/2}\left[\theta\exp\left\{j\omega\int_z^\Delta C(y)|v_1(y)|dy\right\}\right.$$

$$\left. + j\omega\bar{\theta}\int_z^\Delta C(y)v_1^2(y)\ \exp\left\{2j\omega\int_y^\Delta C(s)|v_1(s)|^2ds\right\}dy\ \exp\left\{-j\omega\int_z^\Delta C(y)|v_1(y)|dy\right\}\right]$$

where $\theta = (1+\omega^2Y^2\Delta^2)^{1/2} \,/\, (1+j\omega Y\Delta)$, a complex number of unit modulus.

If it is desired, the reflection coefficient can be computed from the relation

$$R = \left[V(0)C^{1/2}(0) - I(0)\right] \Big/ \left[V(0)C^{1/2}(0) + I(0)\right]. \tag{4.11}$$

As an example of how one might use the formulas to estimate quantities of interest, consider the case of a very thin layer: $\Delta \sim 0$. From (4.10), (4.3) and (4.6) we get roughly (neglecting terms of order Δ^2):

$$V(0) \sim \tilde{V}(\Delta) \,/\, (2Y)^{1/2} \tag{4.12}$$

$$I(0) \sim \tilde{V}(\Delta)\left[Y + j\omega\Delta\left\{C(\Delta) - Y^2\right\}\right] \Big/ (2Y)^{1/2}.$$

If we assume, as is the usual case, that $C(\Delta) = Y^2$, then we have

$$R \sim \left[C^{1/2}(0) - Y\right] \Big/ \left[C^{1/2}(0) + Y\right] + O(\Delta^2). \tag{4.13}$$

Except for terms of order Δ^2, (4.13) is just the expression for the reflection coefficient of two different homogeneous transmission lines joined at the point z = 0, where $1/C^{1/2}(0)$ is the characteristic impedance of the first line and $1/Y$ is the characteristic impedance of the second line. In the particular case where

$$C(0) = C(\Delta),$$

R vanishes except for terms of order Δ^2.

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